Concerning a natural compatibility condition between the action and the renormalized operator product

Olivier de Mirleau
mirleau@wins.uva.nl [1]

Contents

1. Abstract

In this article we note that in a number of situations the operator product and the classical action satisfy a natural compatibility condition. We consider the interest of this condition to be twofold: First, the naturality (functoriality) of the compatibility condition suggests that it be used for geometrical applications of renormalized functional integration. Second, the compatibility can be used as the definition of a category; consideration of this category as the central object of study in quantum field theory seems to have quite some advantages over previously introduced theories of the type "S-matrix theory", "Vertex operator algebras", since this seems to be the only category in which both the action and the expectation values enter, the two being linked roughly speaking by a combination of the Frobenius property and the renormalized Schwinger-Dyson equation.

[1] Department of Mathematics/ University of Amsterdam/ Plantage Muidergracht 24/ 1018 TV Amsterdam/ The Netherlands

2. Introduction.

Functional integrals are a very compact way of expressing expectation values relevant for the description of physical phenomena in terms of an action, but they have a disadvantage, being that they do not contain enough information: The action has to be supplemented with renormalization conditions before expectation values can be determined. This article aims at providing a setting in which this kind of information can be handled more systematically.

This leads us to an interesting axiomatization of renormalized quantum field theory in general, not restricting to a certain action-independent aspect of it like the gluing property, or the presence of a certain symmetry. This axiomatization seems adequate as a setting for quantum field theory because it requires a link between the action and the expectation values, unlike for example S-matrix theory [6, chapter 2-3], the theory of n-point functions satisfying Wightman's axioms [5, chapter 3], Segal's gluing functors [12][13], or the theory of vertex operator algebras [11].

The article is organized as follows: In section 3 we define renormalization and illustrate it with a number of examples. In section 4 we first remind the link between normal ordering and renormalization, and then prove that a natural compatibility condition with the action is satisfied by normal ordering for actions of which the first derivatives constitute a coordinate system. In section 5 we define the category of renormalized volume manifolds, which axiomatize our compatibility condition, and illustrate them with some easy examples. The appendix contains a review of functional integration, partially overlapping with contents of [14].

3. Renormalization.

3.1. **The general idea.** The general idea of renormalization is that if a function $G(\beta, \Lambda)$ of several variables diverges as one of the variables approaches a certain limit, say $\Lambda \to \infty$, then this divergence can sometimes be "compensated" or "removed" by simultaneously adapting the other variables, say $\beta \to -\infty$. I.e. β is replaced by a function of Λ such that $\lim_{\Lambda\to\infty} G(\beta(\Lambda), \Lambda)$ is non-empty.

In general of course this limit will depend on the choice of function $\beta(.)$. One may however ask exactly what set of values can be obtained in this way by varying the function $\beta(.)$, i.e. we may try to specify the set:

$$\cup_{\beta(.)} \lim_{\Lambda\to\infty} G(\beta(\Lambda), \Lambda).$$

This expression can be simplified by eliminating reference to the functions $\beta(.)$: Indeed, if $\mathcal{B}$ is the set of values for β, then such limit points are in the closure $\overline{G(\mathcal{B}, (n, \infty))}$, for any n. Conversely, under general assumptions [2] one may prove that this is equivalent to the previous expression. This leads to the following definition:

3.2. **The definition of renormalization.** We propose the following as a general definition of renormalization; We will illustrate it with examples below:

Definition 3.2.1. *Let three topological spaces be given:*

1. *$\mathcal{B}$, the space of Bare parameters.*
2. *$\mathcal{C}$, the space of Cutoffs.*
3. *$\mathcal{E}$, the space of Expectation values, or of possible Experimental values.*

Let $\Lambda_\infty \in \mathcal{C}$ be a point in the cutoff space, and suppose given a function depending on bare parameters and cutoffs, which is not defined at Λ_∞:

$$G : \mathcal{B} \times (\mathcal{C} - \{\Lambda_\infty\}) \to \mathcal{E}.$$

Then:

1. *We define the limit $\Lambda \to \Lambda_\infty$ of the sets $G(\mathcal{B}, \Lambda)$ as follows:*
$$\lim G := \lim_{\Lambda\to\Lambda_\infty} (G(\mathcal{B}, \Lambda)) := \cap_{\mathcal{U} \text{ neighborhood of } \Lambda_\infty} \overline{G(\mathcal{B}, \mathcal{U} - \{\Lambda_\infty\})}.$$
2. *We say that G is renormalizable iff $\lim G$ is non-empty.* [3]
3. *A set of locally defined functions $\rho_{i\in I} : \mathcal{E} \to \mathbb{R}$ such that their restriction to $\lim G$ is a local bijection $\lim G \to \mathbb{R}^I$ will be called a set of renormalized parameters.*

[2] Let the space $\mathcal{E}$ of 3.2.1 be first countable, $p \in \lim G$, then there are β_n, λ_n such that $G(\beta_n, \lambda_n) \to p$ and $\lambda_n \to \infty$. Restricting to an increasing subsequence of the λ_n's, we may extend it to an invertible $x \mapsto \lambda_x$ with $\lim_{x\to\infty} G(\beta_{\lceil x \rceil}, \lambda_x) = p$, and since λ is invertible this implies that p is a limit point of the first form.

[3] For applications to the description of physical phenomena it is common to strengthen this definition by requiring that the limit be finite-dimensional; The idea is that all experimental data together is roughly represented by a point $e \in \mathcal{E}$, whereas G determines a limiting set in $\mathcal{E}$: A possible description of e would be to just give all its coordinates in $\mathcal{E}$, but since $\mathcal{E}$ is very big this is not a practical way. It turns out that we can describe e in a very satisfactory way by first narrowing it down to be in a finite dimensional limit set of some relatively simple G, and then fixing e by giving a finite number of coordinates of e (the renormalization conditions) along with G. Infinite dimensional limiting sets are not very useful for this purpose since one would have to specify an infinity of experimental results to pinpoint e.

3.3. **Two examples.** We will now consider two examples of renormalization. In the first example, $\mathcal{E}$ will be two-dimensional and one may draw a picture of what is happening. The second example will be less suitable for pictures, but it is an example of renormalizing divergent integrals, which is what we will concentrate on later.

3.3.1. *Low dimensional example of renormalization.* As an illustration, consider the following case:

$$\mathcal{B} := \mathbb{R};\ \ \mathcal{C} := (0,\infty];\ \ \mathcal{E} := \mathbb{R}^2;\ \ G(\beta,\Lambda \neq \infty) := (\beta+\Lambda, 1-\Lambda^{-1}).$$

Note that $G(\beta,\infty)$ is not defined. But if we consider all β's at the same time, then the limit $\Lambda \to \infty$ can be defined:

$$G(\mathbb{R},\Lambda) = (\mathbb{R}+\Lambda, 1-\Lambda^{-1}) = (\mathbb{R}, 1-\Lambda^{-1}) \to (\mathbb{R},1).$$

The sense in which we consider this set to be the limiting set is exactly stated in our general definition: Indeed, we have

$$\cap_{n\in\mathbb{N}}\overline{G(\mathcal{B},(n,\infty))} = \cap_{n\in\mathbb{N}}\overline{\mathbb{R}\times(1-n^{-1},1)} = \cap_{n\in\mathbb{N}}\mathbb{R}\times[1-n^{-1},1] = (\mathbb{R},1).$$

Thus, G above is renormalizable with limiting set $(\mathbb{R},1)$; the first coordinate is a renormalized parameter for G.

We have seen that the divergence $\lim_{\Lambda\to\infty} G(\beta,\Lambda)$ has been removed by taking all β's together. Another common way of saying this is that the divergence has been "absorbed" into the bare parameters in the step $\mathbb{R}+\Lambda = \mathbb{R}$.

Note the relation between the renormalized parameters, and the absorption of the divergence: Indeed, setting $\beta_\rho(\Lambda) := \rho - \Lambda$, we have

$$G(\beta_\rho(\Lambda),\Lambda) = (\beta_\rho(\Lambda)+\Lambda, 1-\Lambda^{-1}) = (\rho, 1-\Lambda^{-1}) \to (\rho,1).$$

I.e. we can also think of the renormalized parameters as parametrizing a set of functions $\beta(.)$ such that the different possible limiting values of G are exactly obtained by substituting these functions $\beta(.)$ for the variable β.

3.3.2. *An example of renormalized divergent integration.* For the next example, set

$$\mathcal{B} := \mathbb{R}^{>0}\times\mathbb{R}^{>0};\ \mathcal{C} := (0,\infty];\ \mathcal{E} := \mathbb{R}^{\mathbb{N}} = Map(\mathbb{N},\mathbb{R}),$$

$$G(\beta_1,\beta_2;\Lambda) := \{n \mapsto \int_0^\Lambda (\beta_1 x)^n(\beta_2 x)dx\}.$$

In other words:

$$G(\beta,\Lambda) = \{n\mapsto \beta_1^n\beta_2\Lambda^{n+2}/(n+2)\} = (\beta_2\Lambda^2/2, \beta_1\beta_2\Lambda^3/3, \beta_1^2\beta_2\Lambda^4/4, ...)$$

The set $G(\mathcal{B},\Lambda)$ is a two-dimensional subset of $\mathbb{R}^{\mathbb{N}}$, which we may try to parametrize with the first two coordinates, which we call ρ_1 and ρ_2. Thus, on the set $G(\mathcal{B},\Lambda)$ we have the following relations:

$$\rho_1 = \beta_2\Lambda^2/2;\ \ \rho_2 = \beta_1\beta_2\Lambda^3/3,$$

so that the ρ's are positive, since the β's are. These relations can be inverted, giving

$$\beta_1 = \frac{3\rho_2}{2\rho_1\Lambda};\ \ \beta_2 = \frac{2\rho_1}{\Lambda^2},$$

which we can in turn substitute in G. Thus, $(\rho_1,\rho_2) \in \mathcal{R} := \mathbb{R}^{>0}\times\mathbb{R}^{>0}$, are renormalized parameters since with $F : \mathcal{R} \to \mathcal{E} = \mathbb{R}^{\mathbb{N}}$:

$$F(\rho_1,\rho_2) := \{n\mapsto (2\rho_1)^{1-n}(3\rho_2)^n/(n+2)\} = (\rho_1,\rho_2,\frac{9\rho_2^2}{8\rho_1},...),$$

we see that for all Λ: $G(\mathcal{B},\Lambda) = F(\mathcal{R})$, and in particular $F(\mathcal{R}) = \lim_{\Lambda\to\infty} G(\mathcal{B},\Lambda)$. Other renormalized parameters are given by (ρ_2,ρ_3), corresponding to the parametrization

$$F(\rho_2,\rho_3) := \{n \mapsto (3\rho_2)^{2-n}(4\rho_3)^{n-1}/(n+2)\} = (\frac{9\rho_2^2}{8\rho_3}, \rho_2, \rho_3, \frac{16\rho_3^2}{15\rho_2}, \ldots).$$

3.4. **Non-generic examples.** In the above example we have seen that $\lim G$ was homeomorphic to the set of bare parameters. This is not true in general: We will now have a look at some examples where the topology or even the dimension of $\mathcal{B} := \mathbb{R}$ and $\mathcal{R} := \lim G$ differ. [4]

First, consider

$$G(\beta,\Lambda) := (\cos(\beta), \sin(\beta)/\Lambda).$$

The limiting set equals $[-1,1]\times\{0\}$, so that $\mathcal{R} \cong [-1,1]$, which has topology different from $\mathcal{B} = \mathbb{R}$. If we replace $\cos(\beta)$ by $\cos(\beta)/\Lambda$, then $\mathcal{R}$ is just a point.

Finally, the next example will be a case where the dimension of $\mathcal{R}$ is bigger than that of $\mathcal{B}$: Indeed, set

$$G(\beta,\Lambda) := e^{-\beta-\Lambda}(\cos(\beta), \sin(\beta)).$$

For fixed Λ this describes a spiral in $\mathbb{R}^2$. We claim that for all n: $G(\mathbb{R},(n,\infty)) = \mathbb{R}^2 - \{0\}$; Indeed, any point in $\mathbb{R}^2 - \{0\}$ can be written as

$$e^{-d}(\cos(\phi), \sin(\phi)) = G(\phi - 2\pi k, d - \phi + 2\pi k) \in G(\mathbb{R},(n,\infty)),$$

if we choose k big enough. Thus, the limiting set is equal to $\mathbb{R}^2$, so that $\mathcal{R}$ has dimension higher than $\mathcal{B}$.

[4] In the context of renormalized functional integration, the case where $\dim(\mathcal{R}) < \dim(\mathcal{B})$ and where "what is left" corresponds to Gaussian expectation values, is referred to as triviality, see for example [10, section 2].

4. Two properties of normal ordering.

4.1. **Reminder on normal ordering.**

4.1.1. *Definition.* Let S be a function on $\mathbb{R}^n$ such that its first derivatives $S_i := \partial_i S$ form a coordinate system. Normal ordering acts on polynomial functions of these coordinates and gives back functions on $\mathbb{R}^n$; It is defined inductively as follows [14]: $N(1) := 1$, and

$$N(S_{i_0}..S_{i_n}) := S_{i_0} N(S_{i_1}..S_{i_n}) - \frac{\partial}{\partial x^{i_0}} N(S_{i_1}..S_{i_n}).$$

4.1.2. *Relation with Schwinger-Dyson equation.* Normal ordering together with the Schwinger-Dyson equation $\langle(\partial_i S)f\rangle = \langle \partial_i f\rangle$ (see appendix) implies

$$\langle N((\partial_i S)f)g\rangle = \langle N(f)\partial_i g\rangle.$$

If N is invertible, then $\{I$ satisfies the Schwinger-Dyson equation $\Leftrightarrow I(f) = ZN^{-1}(f)I(1)\}$, where Z denotes the projection of polynomial functions of the S_i's on their constant part, e.g. $Z(3 + aS_1 + bS_1S_5) = 3$.

4.1.3. *Relation with original definition.* Houriet and Kind's original definition [3, Formula 12] using the creation-annihilation decomposition of operator fields $\phi = \phi^- + \phi^+$ amounts to

$$(:1:) := 1;\ (: f\phi :) := \phi^-(: f :) + (: f :)\phi^+.$$

It satisfies properties analogous to those of N if we replace the operation ∂_i with the commutator $[\phi^+, .]$:

1. $(: \phi_1..\phi_n :) = (: \phi_{\sigma(1)}..\phi_{\sigma(n)} :)$.
2. $(: \phi f :) = \phi(: f :) - [\phi^+, (: f :)]$.
3. $\langle 0|(: \phi f :)g|0\rangle = \langle 0|(: f :)[\phi^+, g]|0\rangle$,

(see [14, Appendix A.4] for more details). This motivates the use of the name normal ordering for N.

4.1.4. *Relation with substraction of singularities.* In infinite dimensions, the partial derivatives ∂_i get replaced by functional derivatives, and solutions of the corresponding Schwinger-Dyson equation exhibit so-called UV-divergences, i.e. short distance singularites as in $\langle\phi(x)\phi(y)\rangle = K|x-y|^{2-D}$. This makes it impossible to take expectation values of say $\phi^2(x)$. However, normal ordering turns out to be a way to substract one singularity from the other such as to leave something finite: Let us restrict the attention to Gaussian normal ordering; then in infinite dimensions we have $N(\phi(x)\phi(y)) = \phi(x)\phi(y) - \langle\phi(x)\phi(y)\rangle$, and we see that $N(\phi^2(x))$ is only defined if the two-point function singularity is regulated, say by some cutoff parameter Λ; N will in general depend on Λ and the particular regulator; The inductive definition now amounts to:

$$N_\Lambda(1) := 1;\ \ N_\Lambda(\phi^{n+1}(x)) := \phi(x)N_\Lambda(\phi^n(x)) - n\langle\phi(x)\phi(x)\rangle_\Lambda N_\Lambda(\phi^{n-1}(x)).$$

The central statement relating N to the substraction of short distance singularities is that $\langle\prod_{j=1}^n N(e^{p_j\phi(x_j)})\rangle = \prod_{i<j} e^{p_ip_j\langle\phi(x_i)\phi(x_j)\rangle}$, which is finite for $x_i \neq x_j$. By taking derivatives w.r.t. p we see that expectation values of products of expressions like $N(\phi^n(x))$ are finite, i.e. we see that normal ordering exactly substracts singularities coming from $\phi^n(x)$. Furthermore, note that the final result of this substraction is independent of the particular regulator used as the cutoff is removed.

4.2. **Property one: Gaussian normal ordering by renormalization.** In this section we will further illustrate the renormalization procedure by showing how Gaussian normal ordering is a case of renormalization. As we have presented it above, the limit $\Lambda \to \infty$ cannot be renormalized since we have not specified bare parameters in which to absorb the divergences. This is what is done below:

Definition 4.2.1. *Define spaces $\mathcal{B}$ and $\mathcal{E}$ as follows:*

1. *The set of bare parameters $\mathcal{B}$ will be the space of infinite triangular matrices with real entries: $\beta = (\beta_{nm})_{n=1..\infty, m=0..n-1}$.*
2. *To every bare matrix β is associated a linear operator R on polynomials of one variable x, as follows:*
$$R_\beta(x^n) := x^n + \sum_{i=0}^{n-1} \beta_{ni} x^i.$$
3. *Let $\mathcal{E}$ be the following space of arrays of functions:*
$$\mathcal{E} := \{E = (E^{n_1,..,n_k})_{n_i=1..\infty, k=0..\infty} | E^{n_1,..,n_k} : (\mathbb{R}^D)^k \to \mathbb{R}\}.$$
4. *With $\mathcal{C} := (0, \infty]$, let $G : \mathcal{B} \times (\mathcal{C} - \infty) \to \mathcal{E}$ be the map which to bare matrix β and cutoff Λ associates the array of functions $G^{n_1,..,n_k}$ given by*
$$G^{n_1,..,n_k}(x_1, ..., x_k) := \langle R_\beta(\phi^{n_1}(x_1))...R_\beta(\phi^{n_k}(x_k))\rangle_\Lambda.$$
5. *Finally, for $n = 1...\infty$ and $m = 0...n$, define functions $\rho_{nm} : \mathcal{E} \to \mathbb{R}$ by*
$$\rho_{nm}(E) := E^{n,1,..,1}(0, \overbrace{0, ..., 0}^{m \ times}),$$
i.e. we have
$$\rho_{nm}(G(\beta, \Lambda)) = \langle R_\beta(\phi^n(0)) \overbrace{R(\phi(0))...R(\phi(0))}^{m \ times}\rangle.$$

The precise sense in which normal ordering is related to renormalization is the following:

Theorem 4.2.2. *The above G is renormalizable; More specifically, there is a limit point satisfying the renormalization condition*
$$\rho_{(n+1),m} = \sum_{i=1}^{m} \rho_{11} \rho_{n,(m-1)}.$$
It is reached by defining the Λ-dependence of the bare parameters to be given by normal ordering: $R_{\beta(\Lambda)} := N_\Lambda$.

Proof

If $R_{\beta(\Lambda)} := N_\Lambda$, then the renormalization condition is satisfied for all Λ since normal ordering satisfies
$$\langle N(\phi^{n+1}(0)) \overbrace{N(\phi(0))...N(\phi(0))}^{m \ times}\rangle$$
$$= \sum_{i=1}^{m} \langle N(\phi(0)) N(\phi(0))\rangle \langle N(\phi^n(0)) \overbrace{N(\phi(0))...N(\phi(0))}^{m-1 \ times}\rangle.$$
G is renormalizable because the cutoff-limit of expectation values of normal ordered integrands exists.

□

4.3. **Property two: Natural compatibility with the action.**

4.3.1. *Normal ordering of vectorfields.* The defining formula $N((\partial_i S)f) = (\partial_i S)N(f) - \partial_i N(f)$ has a number of unsatisfactory features:

1. The transition between x^i's and $\partial_i S$'s may not be invertible; It might therefore be that some functions cannot be written as a function of $\partial_i S$'s.
2. The definition needs the choice of vectorfields ∂_i; On more general manifolds the choice of a subalgebra of vectorfields is needed. One would like to have a more natural notion of substracting short distance singularities, not involving an a priori choice of vectorfields.

These considerations, and the hope of simplifying a number of formulae have led us to introduce normal ordering of vectorfields by $N(X^i\partial_i) := N(X^i)\partial_i$: The combination of N, S and ∇_S [5] then satisfies the following attractive compatibility condition: (We will prove it in a moment):

$$\nabla_S(N(X)) = -N(X(S)).$$

Indeed, this equation has the following features:

1. The vectorfields X are not restricted to be of the form ∂_i. In other words, the naturality of this expression is higher than the seperate defining formulae of $N(f)$ and $N(X)$. This is the naturality that we are referring to in the title of this paper.
2. If we know N and S then we know ∇_S, i.e. it makes the relation between divergence ∇ and regulator N explicit once the action is known. Therefore, if we require this compatibility to hold outside our restricted class of actions, then in general for nonlinear targetspaces, the problem of regularizing composite operators and that of defining ∇_S are the same.
3. The Schwinger-Dyson equation can now be stated without ∇_S, instead using only N: $[N(X(S))f] = [N(X)(f)]$.

With some hindsight, we can reformulate the above as follows: There are two occasions on which the sole datum of an action S is not enough to specify expectation values:

1. S does not determine what we mean by $\langle R(\phi^2(x))\phi(y)\phi(z)\rangle$, since R as such is not determined by the action alone. It is only fixed after extra conditions are given, one of these conditions for example leading to $R = N$, normal ordering.
2. If target space is nonlinear, then $\langle . \rangle$ is undetermined by an action, since the very statement of the Schwinger-Dyson equation needs a divergence ∇, not an action S. (See appendix A.2.4).

We now see explicitly that these two matters are related since the compatibility condition relates N and ∇_S. ¿From a more general standpoint this can be understood as follows: In the case of nonlinear targetspaces, there is no distinction between expressions that are linear and those are not. But non-linear arguments like $\phi^2(x)$ are exactly those that need regularizing. Thus, the nonlinear targetspaces are expected to need ultraviolet regulators right from the start, so that at fixed action S one can understand that some information contained in ∇_0 is reflected in information contained in the choice of ultraviolet regulators.

[5] With $\mu_S := e^{-S}dx^1...dx^n$, ∇_S is the divergence of μ_S: $\nabla_S(X)\mu_S := L_X\mu_S$.

Let us now prove the compatibility relation: (It can of course by definition only be proved for the restricted class of actions where $\partial_i S$'s are coordinates, but the class seems flexible enough to want to extend the compatibility to all actions).

Theorem 4.3.1. *Let $\nabla_0(X) = \partial_i X^i$, $\nabla_S(X) = \nabla_0(X) - X(S)$, and let N map vectorfields to vectorfields and functions to functions such that $N(1) = 1$. Assume the first derivatives $\partial_i S$ generate the algebra of functions f under consideration. Then the following three points are equivalent:*

1.
$$\nabla_S(N(X)) = -N(X(S)),$$
$$N(X^i \partial_i) = N(X^i)\partial_i.$$

2.
$$N((\partial_i S)f) = (\partial_i S)N(f) - \partial_i N(f),$$
$$N(\partial_i) = \partial_i; \;\; N((\partial_i S)X) = (\partial_i S)N(X) - [\partial_i, N(X)].$$

3.
$$N((\partial_i S)f) = (\partial_i S)N(f) - \partial_i N(f),$$
$$N(X^i \partial_i) = N(X^i)\partial_i.$$

Proof

1. $(2 \Rightarrow 3)$. Induction on $|X^i|$; Assume true for X^i, we will prove it for $(\partial_j S)X^i$:
$$N((\partial_j S)X^i \partial_i) = (\partial_j S)N(X^i \partial_i) - [\partial_j, N(X^i \partial_i)]$$
$$= (\partial_j S)N(X^i)\partial_i - [\partial_j, N(X^i)\partial_i]$$
$$= \{(\partial_j S)N(X^i) - \partial_j N(X^i)\}\partial_i = N((\partial_j S)X^i)\partial_i.$$
2. $(3 \Rightarrow 2)$:
$$N((\partial_i S)X) = N((\partial_i S)X^j \partial_j)$$
$$= N((\partial_i S)X^j)\partial_j = (\partial_i S)N(X^j)\partial_j - (\partial_i N(X^j))\partial_j$$
$$= (\partial_i S)N(X) - [\partial_i, N(X^j)\partial_j] = (\partial_i S)N(X) - [\partial_i, N(X)].$$
3. $(2 + 3 \Rightarrow 1)$: Induction on $|X|$, first $|X| = 0$:
$$\nabla_S(N(\partial_i)) = \nabla_S(\partial_i) = -\partial_i S = -N(\partial_i S).$$
Assume true for X, we will prove it for $(\partial_j S)X$:
$$\nabla_S(N((\partial_j S)X)) + N((\partial_j S)X(S))$$
$$= \nabla_S((\partial_j S)N(X) - [\partial_j, N(X)]) + (\partial_j S)N(X(S)) - \partial_j N(X(S))$$
$$= (\partial_j S)\nabla_S(N(X))_1 - \partial_j \nabla_S(N(X))_2 + (\partial_j S)N(X(S))_1 - \partial_j N(X(S))_2$$
$$+ N(X)(\partial_j S)_3 + N(X)\nabla_S(\partial_j)_3 = 0.$$
4. $(1 \Rightarrow 3)$. Induction on $|f|$. $f = 1$:
$$N(\partial_i S) = -\nabla_S(N(\partial_i)) = -\nabla_S(\partial_i) = \partial_i S.$$
Suppose the relation proved for f, we will prove it for $(\partial_j S)f$:
$$N((\partial_j S)f(\partial_i S)) = -\nabla_S(N((\partial_j S)f\partial_i)) = -\nabla_S(N((\partial_j S)f)\partial_i)$$
$$= -N((\partial_j S)f)\nabla_S(\partial_i) - \partial_i N((\partial_j S)f)$$
$$= (\partial_i S)N((\partial_j S)f) - \partial_i N((\partial_j S)f).$$

$\square$

These considerations lead us to the following definition of compatibility between regulators and the action S, whatever type S may be of:

Definition 4.3.2. *Let A be a symmetric associative algebra with unit, $L := Der(A)$. Let $S \in A$. Let $N : A \to A$, and $N : L \to L$ be invertible. We say that (S, N) is compatible iff ∇ defined by $\nabla(N(X)) = -N(X(S))$ is a divergence operator, i.e. satisfies*

1. $\nabla([X, Y]) = X(\nabla(Y)) - Y(\nabla(X))$.
2. $\nabla(fX) = X(f) + f\nabla(X)$.

The associated Schwinger-Dyson equation reads: $[N(X(S))f] = [N(X)(f)]$, the idea being that this is the nonlinear generalization <u>and</u> the ultraviolet regularization of $[(\partial_i S)f] = [\partial_i(f)]$. In the rest of this section N will be assumed to satisfy compatibility with S.

Note that the compatibility condition as such does not determine N, since we dropped the extra requirement $N(X) = N(X^i)\partial_i$ that fixes N. The idea is that the condition $\nabla_S(N(X)) = -N(X(S))$ is very general, whereas an extra condition like $N(X) = N(X^i)\partial_i$ is special for the situation under consideration. I.e. the choice of vectorfields ∂_i is fine when working on linear spaces, but on nonlinear spaces, other types of renormalization conditions will have to be imposed if structure invariance is to be preserved. [6]

4.3.2. *Renormalized algebraic operations and the renormalized Schwinger-Dyson equation.* Finally let us push the abstraction a little further, in order to remove one last bad property of normal ordering, which is that it is only defined at finite cutoff: $N_\Lambda(\phi^2(x))$ diverges in the cutoff limit. This property can be circumvented, by restricting the attention to expressions of the type $N^{-1}(N(A(x))N(B(y)))$ at $x \neq y$, since e.g. in the Gaussian case

$$N^{-1}(N(e^{\phi(x)})N(e^{\phi(y)})) = e^{\langle\phi(x)\phi(y)\rangle}e^{\phi(x)+\phi(y)},$$

which is well defined. Thus, we are led to define renormalized algebraic operations

$$f \cdot_r g := N^{-1}(N(f)N(g)),\ f \cdot_r X := N^{-1}(N(f)N(X)),$$

and analogously for $X \cdot_r f$ and $[X,_r Y]$, and renormalized expectation values

$$\langle \mathcal{O}_1, ..., \mathcal{O}_n \rangle_r := \langle N(\mathcal{O}_1)...N(\mathcal{O}_n) \rangle,$$

and forgetting completely about N, by just working with these new structures only. Note that the new structure will also satisfy associativitly, Jacobi, etc., since it is the pullback of such operations by an invertible map. Since we have $\langle \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3 \rangle = \langle \mathcal{O}_1 \cdot_r \mathcal{O}_2, \mathcal{O}_3 \rangle$, we will refer to $f \cdot_r g$ as the operator product [8, Section II] of f and g. Furthermore:

Theorem 4.3.3. *The compatibility condition of N with S implies:*

$$X \cdot_r (YS) - Y \cdot_r (XS) = [X,_r Y]S,$$

$$(f \cdot_r X)S = f \cdot_r (XS) - X \cdot_r f.$$

The Schwinger-Dyson equation can be written as:

$$\langle XS, \mathcal{O}_1, ..., \mathcal{O}_n \rangle_r = \sum_{i=1}^{n} \langle \mathcal{O}_1, .., X \cdot_r \mathcal{O}_i, .., \mathcal{O}_n \rangle_r.$$

---

[6]For example if targetspace is a Riemannian manifold, a probable way would be to make N act diagonally on the eigenfunctions of the Laplacian, by analogy with $N(e^{px}) = e^{p^2/2}e^{px}$.

Proof

1.
$$\begin{gathered}N(X\cdot_r(YS)-Y\cdot_r(XS))=N(X)N(YS)-N(Y)N(XS)\\=-N(X)\nabla(N(Y))+N(Y)\nabla(N(X))=-\nabla([N(X),N(Y)])\\=-\nabla(N([X,_r Y]))=N([X,_r Y]S).\end{gathered}$$

2.
$$\begin{gathered}N((f\cdot_r X)S)=-\nabla(N(f\cdot_r X))=-\nabla(N(f)N(X))\\=-N(f)\nabla(N(X))-N(X)(N(f))\\=N(f)N(XS)-N(X\cdot_r f)=N(f\cdot_r(XS)-X\cdot_r f).\end{gathered}$$

3. Finally, for the Schwinger-Dyson equation:
$$\begin{gathered}\langle XS,\mathcal{O}_1\rangle_r=\langle N(XS)N(\mathcal{O}_1)\rangle=\langle N(X)N(\mathcal{O}_1)\rangle\\=\langle N(X\cdot_r\mathcal{O}_1)\rangle=\langle X\cdot_r\mathcal{O}_1\rangle_r.\end{gathered}$$

□

This motivates us to study the category introduced in section 5.

5. Compatibility of operator product and action as a starting point.

5.1. **Renormalized volume manifolds.** (For a review of volume manifolds, see appendix A.2.4).

Definition 5.1.1. *Let A be a symmetric associative algebra with unit, $L := Der(A)$, with corresponding operations fg, fX, Xf and $[X,Y]$. By a renormalized structure on A we mean the datum of $S \in A$, together with extra multiplications $f._r g$, $f._r X$, $X._r f$ and $[X,_r Y]$, such that*

1. *The $._r$-operations induce $L = Der(A, ._r)$, i.e. the derivations of the multiplication $(f,g) \mapsto f._r g$ are exactly given by the operations $f \mapsto X._r f$ as $X \in L$.*
2. *Both structures have the same unit: $1f = f = 1._r f$.*
3. *The two algebraic structures and the action S are compatible in the sense that:*

$$X._r(YS) - Y._r(XS) = [X,_r Y]S,$$
$$(f._r X)S = f._r(XS) - X._r f.$$

A together with the renormalized structure will be called a renormalized volume manifold. For symmetric linear maps $\langle ,,, \rangle_r : A^{\otimes n} \to \mathbb{R}$ we will be interested in the following properties:

1. *$\langle .,. \rangle_r : A^{\otimes 2} \to \mathbb{R}$ is a positive nondegenerate form. (Positivity).*
2. *$\langle \mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_n \rangle_r = \langle \mathcal{O}_1 ._r \mathcal{O}_2, .., \mathcal{O}_n \rangle_r$, (Frobenius).* [7]
3. *$\langle XS, \mathcal{O}_1, ..., \mathcal{O}_n \rangle_r = \sum_{i=1}^n \langle \mathcal{O}_1, .., X._r \mathcal{O}_i, .., \mathcal{O}_n \rangle_r$. (Schwinger-Dyson).*

By substituting $S/\hbar$ for S and taking the limit $\hbar \to 0$, we see that at $\hbar = 0$, we may take both algebraic structures to be equal. In what follows, we will take the usual multiplication to have priority over the renormalized one, i.e. $f._r gh$ means $f._r(gh)$.

5.2. **Example 1: Renormalization conditions for Gaussian integrals.** The following theorem says that in the Gaussian case the renormalized product subjected to the renormalization condition $f\partial_i ._r g = f._r \partial_i g$ equals $e^{px}._r e^{qx} = e^{\hbar pq} e^{(p+q)x}$:

Theorem 5.2.1. *Let A be the polynomial functions on $\mathbb{R}^D$, L the polynomial vectorfields, and let $S = g_{ij}x^i x^j/2$. Then there is exactly one renormalized operation satisfying the renormalization condition $f\partial_i ._r g = f._r \partial_i g$, and there is exactly one solution to the corresponding Frobenius-Schwinger-Dyson equation.*

Proof
(We use formal power series e^{px} for neater expressions). Indeed, let us first prove that $e^{px}._r e^{qx} = K(p,q)e^{(p+q)x}$ for some K:

$$\partial_i(e^{px}._r e^{qx}) = \partial_i ._r(e^{px}._r e^{qx}) = (\partial_i ._r e^{px})._r e^{qx} + e^{px}._r(\partial_i ._r e^{qx})$$
$$= (\partial_i e^{px})._r e^{qx} + e^{px}._r(\partial_i e^{qx}) = (p+q)_i e^{px}._r e^{qx},$$

which implies the above form. Next note that $(f._r g)\partial = f._r(g\partial)$ which is true since both sides have the same renormalized action:

$$[(f._r g)\partial]._r h = (f._r g)._r \partial h = f._r(g._r \partial h) = f._r(g\partial._r h) = (f._r g\partial)._r h.$$

We may use this identity to show that $K(p,q) = e^{pq}$:

$$(\partial_q K(p,q))e^{(p+q)x} =_{(Leibniz)} \partial_q(K(p,q)e^{(p+q)x}) - Kxe^{(p+q)x}$$
$$=_{(Definition\ of\ K)} \partial_q(e^{px}._r e^{qx}) - (e^{px}._r e^{qx})\partial S$$

[7] A Frobenius algebra is a symmetric associative algebra with metric such that $(a, bc) = (ab, c)$.

$$=_{(Previous\ identity)} e^{px}\cdot_r(e^{qx}\partial S) - (e^{px}\cdot_r e^{qx}\partial)S$$
$$=_{(Compatibility\ with\ S)} e^{qx}\partial\cdot_r e^{px} = e^{qx}\cdot_r\partial e^{px} = pK(p,q)e^{(p+q)x}.$$

So that $K(p,q) = e^{pq}$. This in turn determines $X\cdot_r f$ by using the renormalization condition, and $f\cdot_r X$ by the identity $f\cdot_r(g\partial_i) = (f\cdot_r g)\partial_i$.

We will now show that indeed $L = Der(A, \cdot_r)$. By taking derivatives, $x^i\cdot_r x^j = g^{ij} + x^i x^j$, and A is $\cdot_r$-generated by the x^i's. Therefore any derivation D_r of $\cdot_r$ equals $D_r(x^i)\cdot_r\partial_i\cdot_r$, which is indeed of the form $X\cdot_r$ with $X \in L$, since $D_r(x^i)\cdot_r\partial_i$ is in L. Finally, $[X,_r Y]$ is determined by $[X,_r Y]\cdot_r g = X\cdot_r Y\cdot_r g - Y\cdot_r X\cdot_r g$.

As for the expectation values, they follow from the operator product, but can also be computed directly:

$$\begin{aligned}\partial_{p_0^i}\langle e^{p_0x}, .., e^{p_nx}\rangle_r &= \langle e^{p_0x}\partial_{x^i}S, e^{p_1x}, .., e^{p_nx}\rangle_r \\ &= \sum_{j=1}^n \langle e^{p_1x}, .., e^{p_0x}\partial_i\cdot_r e^{p_jx}, ..e^{p_nx}\rangle_r \\ &= \sum_{j=1}^n \langle e^{p_1x}, .., e^{p_0x}\cdot_r\partial_i e^{p_jx}, ..e^{p_nx}\rangle_r \\ &= (p_1 + ... + p_n)_i\langle e^{p_0x}, .., e^{p_nx}\rangle_r.\end{aligned}$$

This determines $\langle . \rangle_r$

□

5.3. **Example 2: Renormalization conditions for Schroer's Lagrangian.** Schroer's Lagrangian [4, Formulae 24,30] is given by

$$\mathcal{L}^\lambda(\psi, \bar\psi, \phi) := \mathcal{L}_0(\phi) + [\bar\psi(\gamma^\mu\partial_\mu - im)\psi + i\lambda\bar\psi\gamma^\mu\psi\partial_\mu\phi]dx^1..dx^D,$$

where $\mathcal{L}_0(\phi)$ is some solved Lagrangian for ϕ, say quadratic. For ease of computation we consider such a Lagrangian to be an element of the symmetric algebra on even symbols $\phi(x)$ and odd symbols $\psi^A(x), \bar\psi^A(x)$, and their derivatives. γ_μ is a symmetric Dirac matrix $\gamma_\mu^{AB} = \gamma_\mu^{BA}$. Schoer's interest in this Lagrangian was raised by the fact that it is nonrenormalizable by powercounting, but exactly solvable by the identity $\mathcal{L}^\lambda(\psi, \bar\psi, \phi) = \mathcal{L}^0(e^{i\lambda\phi}\psi, e^{-i\lambda\phi}\bar\psi, \phi)$.

5.3.1. *Naive functional integral solution.* In naive functional integral notation, the solution is easy: If $\mathcal{O}$ is a function of $(\psi, \bar\psi, \phi)$, then

$$\begin{aligned}\langle\mathcal{O}\rangle^\lambda &= \int D\phi D\psi D\bar\psi e^{-S^\lambda(\psi,\bar\psi,\phi)}\mathcal{O}(\psi,\bar\psi,\phi) = \int D\phi D\psi D\bar\psi e^{-S^0(e^{i\lambda\phi}\psi, e^{-i\lambda\phi}\bar\psi,\phi)}\mathcal{O}(\psi,\bar\psi,\phi) \\ &= \int D\phi De^{-i\lambda\phi}\psi De^{i\lambda\phi}\bar\psi e^{-S^0(\psi,\bar\psi,\phi)}\mathcal{O}(e^{-i\lambda\phi}\psi, e^{i\lambda\phi}\bar\psi,\phi) \\ &= \int D\phi D\psi D\bar\psi e^{-S^0(\psi,\bar\psi,\phi)}\mathcal{O}(e^{-i\lambda\phi}\psi, e^{i\lambda\phi}\bar\psi,\phi) = \langle\mathcal{O}(e^{-i\lambda\phi}., e^{i\lambda\phi}., .)\rangle^0,\end{aligned}$$

which e.g. for the two-point function reads

$$\langle\psi^A(x)\bar\psi^B(y)\rangle^\lambda = \langle e^{-i\lambda\phi(x)}\psi^A(x)e^{i\lambda\phi(y)}\bar\psi^B(y)\rangle^0,$$

which is however nonsense since it contains unregularized composite operators. Schroer's expectation value is the following modification of this expression in case $\mathcal{L}_0$ is quadratic:

$$\langle\psi^A(x)\bar\psi^B(y)\rangle^\lambda_{Schroer} := \langle N(e^{-i\lambda\phi(x)})\psi^A(x)N(e^{i\lambda\phi(y)})\bar\psi^B(y)\rangle^0$$

$$= e^{\lambda^2\langle\phi(x)\phi(y)\rangle}\langle\psi^A(x)\bar\psi^B(y)\rangle^0,$$

where N is normal ordering for the Lagrangian $\mathcal{L}_0$. Or more explicitly if $\mathcal{L}_0(\phi) := (\partial_\mu\phi)^2/2$, with $k_D := Vol(S^{D-1})^{-1}$:

$$\langle\psi^A(x)\bar\psi^B(y)\rangle^\lambda_{Schroer} = \langle\psi^A(x)\bar\psi^B(y)\rangle^0 \times \begin{cases} \exp\{k_D\lambda^2|x-y|^{2-D}\} & D \neq 2 \\ |x-y|^{-k_D\lambda^2} & D = 2 \end{cases}$$

where we see an essential singularity in $|x-y|$ as soon as $D > 2$, which is in agreement with powercounting. [8]

5.3.2. *More sensible solution.* The above formulation is unsatisfactory in that the problem was defined during its own solution. With the notions that we have introduced in the previous sections, it will now not be very difficult to state the exact renormalization conditions which single out Schroer's solution as a solution to the Frobenius-Schwinger-Dyson equation:

Theorem 5.3.1. *There is exactly one solution of the Frobenius-Schwinger-Dyson equation for Schroer's Lagrangian $\mathcal{L}^\lambda$, satisfying the renormalization condition $\forall_{f,g}\ fX._r g = f._r Xg$, for X the following derivations:*

$$e^{i\lambda\phi}\frac{\delta}{\delta\psi^A};\ e^{-i\lambda\phi}\frac{\delta}{\delta\bar\psi^A};\ \ \frac{\delta}{\delta\phi} - i\lambda\{\bar\psi^A\frac{\delta}{\delta\bar\psi^A} - \psi^A\frac{\delta}{\delta\psi^A}\}.$$

Proof

Indeed, we will prove in a moment that under the map $M : \psi \mapsto e^{i\lambda\phi}\psi, \phi \mapsto \phi$ between symbolic algebras, these vectorfields get mapped to $\frac{\delta}{\delta\psi^A}$, etc. Now we already know that there is a unique solution for Gaussian actions of the Frobenius-Schwinger-Dyson equation such that these last vectorfields commute with the $._r$ operation. Thus, pulling everything back by the invertible M, we see that there is

[8] Powercounting [2, Section V] is the order by order in λ analysis of the divergence of the Fourier transform of the expectation values. The result is that the divergence of the Fourier transform becomes worse as the order of λ increases if the so-called dimension of the corresponding nonquadratic term in the Lagrangian is bigger than zero; The bad behaviour criterion for Schroer's Lagrangian reads $0 < [\bar\psi\gamma^\mu\psi\partial_\mu\phi dx^1..dx^D] = 2[\psi] + [\partial_x] + [\phi] + D[dx]$, where $[\partial_x] := -[dx] := 1$, and $[\psi], [\phi]$ are defined as *half* the power of singularity of the unregularized Gaussian *two*-point functions, in this case $[\phi] = \frac{1}{2}(D-2)$, $[\psi] = \frac{1}{2}(D-1)$, giving $0 < (D-1) + 1 + (D/2-1) - D = D/2 - 1$. We can see this directly from the exact solution, since the essential singularity is present for $D > 2$ only. More generally powercounting asserts that regularized Fouriertransformed expectation values of a finite-dimensional linear family of Lagrangians with a term of positive dimension are not renormalizable within this family, even if one includes all possible Lagrangians $\mathcal{L}_i$ with $[\mathcal{L}_i] \leq 0$ in this family (the so-called counterterms: They are designed to possibly absorb divergences in their linear pre-factor, and are restricted such as to not worsen the divergences that one wishes to absorb). Here it is understood that the family should remain renormalizable when using different regulators of the Gaussian two-point functions, like for normal ordering. E.g. if the regulated Gaussian part was just $\phi\Delta(1 + (\Delta/\Lambda)^9)\phi$, then a simple counterterm would be $\phi\Delta^9\phi$, which after renormalization would amount to keeping Λ finite. However the answer would depend on k. Thus, in our language it is better to think of the space of cutoffs to be very big in order to include all k's, and the space $\mathcal{B}$ to be the linear family of Lagrangians mentioned above.

The powercounting criterion seems to be really about the renormalizability of the Fourier transform only; From Schroer's example we see that it doesn't give us any information on the existence of the functional integral in position space. This asymmetry originates in the fact that a Lagrangian which is local in x's is non-local in p's. We may however use the momentum powercounting tot draw conclusions on the type of $|x-y|$ singularities that are to be expected, even if the theory is not exactly solvable. (Assuming that the expectation values exist in x-space).

a unique solution of the Schroer Lagrangian satisfying the pulled-back renormalization conditions. Finally let us prove the statement: We have

$$M(\frac{\delta}{\delta\psi^A})(\mathcal{O}) = M(\frac{\delta}{\delta\psi^A})MM^{-1}(\mathcal{O}) := M(\frac{\delta}{\delta\psi^A}M^{-1}(\mathcal{O})).$$

Therefore,

$$M(\frac{\delta}{\delta\psi^A})(\phi) = M(\frac{\delta}{\delta\psi^A}M^{-1}(\phi)) = 0,$$

$$M(\frac{\delta}{\delta\psi^A(x)})(\psi^B(y)) = M(\frac{\delta}{\delta\psi^A(x)}e^{-i\lambda\phi(y)}\psi^B(y)) = e^{-i\lambda\phi(y)}\delta(x-y)\delta^A_B.$$

I.e. $M(\delta/\delta\psi^A(x)) = e^{-i\lambda\phi(x)}\delta/\delta\psi^A(x)$, leading to the first two formulae; as for the third:

$$M(\frac{\delta}{\delta\phi(x)})(\phi(y)) = \delta(x-y),$$

$$M(\frac{\delta}{\delta\phi(x)})(\psi^A(y)) = M(\frac{\delta}{\delta\phi(x)}e^{-i\lambda\phi(y)}\psi^A(y))$$

$$= -i\lambda\delta(x-y)M(e^{-i\lambda\phi(y)}\psi^A(y)) = -i\lambda\delta(x-y)\psi^A(y).$$

So that $M(\delta/\delta\phi(x)) = \delta/\delta\phi(x) - i\lambda\psi^A(x)\delta/\delta\psi^A(x) + i\lambda\bar{\psi}^A(x)\delta/\delta\bar{\psi}^A(x)$.

□

6. Acknowledgments.

I would like to thank Prof.Dr.R.H.Dijkgraaf and Dr.H.Pijls for supervising my work. Further thanks go to Dr.A.A.Balkema, Dr.E.van den Heuvel, and Prof.Dr.J.Smit for discussions. This work was financially supported by "Samenwerkingsverband Mathematische Fysica van de Stichting FOM en het SMC."

Appendix A. Functional integration.

A.1. **The general idea.**

A.1.1. *Example.* This section is an introduction to the definition of integration over function spaces, possibly infinite dimensional. One may think of such an integral as a limit of finite dimensional integrals, like:

$$\langle x^7x^7x^5x^5\rangle = \lim_{n\to\infty} \frac{\int e^{-(x^1x^1+..+x^nx^n)/2}x^7x^7x^5x^5dx^1..dx^n}{\int e^{-(x^1x^1+..+x^nx^n)/2}dx^1..dx^n} = 1.$$

This limit exists since the expression is independent of n if $n > 7$. Note however that the limit of the unnormalized integrals equals $\lim_{n\to\infty}\sqrt{2\pi}^n = \infty$.

The above formula can be seen as a normalized version of an integral over $\mathbb{R}^{\mathbb{N}}$. We will also be concerned with cases of a "continuum of integral signs", i.e. for example integrals over $\mathbb{R}^{\mathbb{R}}$. Trying to define them in a way analogous to the above would need a discretization of that continuum, which we will avoid by working with defining properties:

A.1.2. *Definition by properties rather than by construction.* Before embarking on the definition of functional integration, let us make a remark on usual integration by noting that $\int_a^b f(x)dx$, in the early days, was defined as $F(b)-F(a)$ where F was a solution of $F' = f$. Only much later was the integral expression defined as the limit of Riemann sums, which at the same time proved existence of an F satisfying $F' = f$, even if one could not find it in terms of standard functions. We will take the same approach to functional integration, stating it as a problem, analogous to $F' = f$, deferring the existence and uniqueness questions of this problem to the indefinite future.

Note however that we cannot just take the defining property to be the infinite dimensional limit of $F' = f$, since

1. This would basically define functional integration to be the <u>unnormalized</u> expression $\lim_{n\to\infty}\int dx^1..dx^n$, which we want to avoid in the above example.
2. We will only be interested in <u>indefinite</u> integrals, so that we will have no use of the values $F(x)$ other than $F(\infty) - F(-\infty)$.
3. <u>Gaussian</u> integrals (corresponding to the description of the least complicated physical situations) would in this way not be particularly easy since the primitive of e^{-x^2} is hard to find, and finally,
4. Since the integrals have to be normalized, the interest is in the comparison of <u>different integrands</u>, say $\langle 1\rangle$ and $\langle x^2x^2\rangle$, not the integral of one single integrand.

To that end we will concentrate on the properties up to normalization of the linear map $f \mapsto [f]$ below, rather than on the function $F(a) := \int_0^a e^{-S}dx$.

A.2. **The finite dimensional Schwinger-Dyson equation.**

A.2.1. *The Schwinger-Dyson equation on* $\mathbb{R}^n$. For fixed $S : \mathbb{R}^n \to \mathbb{R}$, consider the following linear functional:

$$f \mapsto [f] := \int_{\mathbb{R}^n} fe^{-S}dx^1..dx^n,$$

where f and S are restricted such that it is well defined, and such that upon partial integration boundary terms are zero. In that case the functional satisfies three properties:

1. $f > 0 \Rightarrow [f] > 0$. (Positivity).
2. $[1] = \int e^{-S} dx^1..dx^n$. (Normalization).
3. $\forall_{i,f}\ [\partial_i(S)f] = [\partial_i f]$. (Schwinger-Dyson equation).

Indeed, for the last point:

$$0 = \int \frac{\partial}{\partial x^i}(e^{-S}f)dx^1...dx^n = \int e^{-S}(-\partial_i(S)f + \partial_i f)dx^1...dx^n = [\partial_i f] - [f\partial_i S].$$

The interest of this equation is that it can be generalized to infinite dimensions, by substituting functional derivatives for the partial derivatives [1, formula 45]. We set $\langle f\rangle := [f]/[1]$.

A.2.2. *Uniqueness argument.* An important remark to be made on the Schwinger-Dyson equation combined with positivity and normalization is that in finite dimensions it allows for one solution at most;

Theorem A.2.1. [9] *Let S be a polynomial on $\mathbb{R}$ such that $\int_{\mathbb{R}} e^{-S(x)}dx$ converges. Let $C^\infty_{Pol} := \{f \in C^\infty(\mathbb{R},\mathbb{R}) : \exists_{Polynomial\ p}|f| < p\}$, and let $\langle . \rangle : C^\infty_{Pol} \to \mathbb{R}$ be a normalized linear positive functional satisfying $\langle S'f\rangle = \langle f'\rangle$. For $f \in C^\infty_{Pol}$, let $[f] := \int fe^{-S}dx/\int e^{-S}dx$. Then $\forall_{f\in C^\infty_{Pol}}\ \langle f\rangle = [f]$.*

Proof

1. First we prove that there is a K such that $\forall_{f\in C^\infty_c}\ \langle f\rangle = K[f]$, where C^∞_c denotes the C^∞-functions with compact support. By Riesz' representation theorem there is a measure $d\mu$ on $\mathbb{R}$ such that for all $f \in C^\infty_c$: $\langle f\rangle = \int f d\mu$.
   We claim that $\int_{\mathbb{R}} d\mu$ and $\forall_{t\in\mathbb{R}}\tilde{S}(t) := \int_t^\infty S'd\mu$ exist. We will then show using the Schwinger-Dyson equation that $d\mu$ satisfies an identity involving $\tilde{S}$.
   (a) We have $\int_{\mathbb{R}} d\mu = \sup_{f\in C^\infty_c; f\leq 1}\int f d\mu = \sup_{f\in C^\infty_c; f\leq 1}\langle f\rangle \leq \langle 1\rangle = 1$, which proves the first existence.
   (b) Next, if $f \geq 0$ has compact support and $f_\lambda(x) := f(x-\lambda)$, then by dominated convergence $\lim_{\lambda\to\infty}\int f_\lambda d\mu = 0$. Using this, we now prove that $\int_t^\infty S'd\mu$ exists: Indeed we may assume that $S' > 0$ on $[t-\epsilon, \infty)$, so that
   $$\int_t^\infty S'd\mu = \sup_{0\leq f\leq 1;\ f\in C^\infty_c}\int_t^\infty S'f d\mu < sup_\lambda \int_{t-\epsilon}^\infty S'g_\lambda d\mu,$$
   where g_λ are more and more stretched bump functions as λ increases: $supp(g_\lambda) = [t-\epsilon, \lambda+\epsilon]$, and $g_\lambda([t,\lambda]) = 1$.
   $$= \lim_{\lambda\to\infty}\int_{t-\epsilon}^\infty S'g_\lambda d\mu = \lim_{\lambda\to\infty}\langle S'g_\lambda\rangle = \lim_{\lambda\to\infty}\langle g'_\lambda\rangle = \lim_{\lambda\to\infty}\int g'_\lambda d\mu,$$
   which exists.

[9] I thank Dr.A.A.Balkema and Dr.E.van den Heuvel for providing me with the main ideas in this proof. A faster but not very rigourous argument goes as follows: Since $\langle . \rangle$ is positive, one expects it to be given by a positive weight e^{-P}: $\langle f\rangle = \int fe^{-P}dx$. Since $\langle . \rangle$ now satisfies the Schwinger-Dyson equation for both S and P, we have $\forall_f\langle\partial_i(S-P)f\rangle = 0$, which by positivity gives $\partial_i(S) = \partial_i(P)$, or $P = S + c$, so that $\langle f\rangle = K.\int fe^{-S}dx$, so that $\langle . \rangle$ is determined up to a positive scalar, which is in turn determined by the normalization condition.

Let us now derive the identity for $d\mu$ which follows from the Schwinger-Dyson equation: For $f \in C_c^\infty$:

$$\int_{-\infty}^{\infty} f' d\mu = \langle f' \rangle = \langle S' f \rangle = \int_{-\infty}^{\infty} S' f d\mu$$

$$= \int_{x=-\infty}^{\infty} S'(x) \int_{t=-\infty}^{x} f'(t) dt d\mu(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x,t) dt d\mu(x).$$

where $\phi(x,t) := S'(x) f'(t)(x \geq t)$. We claim that we can interchange the integrals: By Fubini it suffices to check that the repeated integral of $|\phi|$ exists. Indeed we already know that $G(t) := \int_t^\infty |S'|(x) d\mu(x) = \int_{-\infty}^\infty |S'(x)|(x \geq t) d\mu(x)$ exists. Therefore $\int \int |\phi| d\mu(x) dt = \int |f'| G dt < \infty$, since $|f'|$ has compact support. Therefore, interchanging the integrals:

$$= \int_{t=-\infty}^{\infty} [\int_{x=t}^{\infty} S'(x) d\mu(x)] f'(t) dt = \int_{t=-\infty}^{\infty} \tilde{S}(t) f'(t) dt.$$

I.e. the functional $I(g) := \int g(d\mu - \tilde{S} dx) : C_c^\infty \to \mathbb{R}$ satisfies $I(C_c^{\infty\prime}) = 0$, i.e. is a linear function on $C_c^\infty / C_c^{\infty\prime}$, which is one-dimensional. The dual of $C_c^\infty / C_c^{\infty\prime}$ is therefore also one-dimensional, and it is generated by $g \mapsto \int g dx$. Therefore:

$$\exists_k \forall_{g \in C_c^\infty} \int g d\mu = \int g(\tilde{S} + k) dx =: \int g d\nu.$$

This implies $d\mu = d\nu$ if $d\mu(K)$ and $d\nu(K)$ are finite for compact K's. Indeed if $f \in C_c^\infty$, $0 \leq f \leq 1$, $f|_K = 1$ then $d\mu(K), d\nu(K) \leq \int_{\mathbb{R}} f d\nu = \int_{\mathbb{R}} f d\mu = \langle f \rangle \in \mathbb{R}$. Therefore $d\mu = (\tilde{S} + k) dx$, so that by the definition of $\tilde{S}$:

$$\tilde{S}(t) = \int_t^\infty S'(x)(\tilde{S}(x) + k) dx,$$

so that $\tilde{S}$ is differentiable, and $-(\tilde{S} + k)' = S'(\tilde{S} + k)$, giving $\tilde{S} + k = \tilde{k} e^{-S}$, so that $d\mu = \tilde{k} e^{-S} dx$.

2. Next let us prove that $K = 1$. To that end we have to fit the result $\langle f \rangle = K[f]$ for f of compact support with $\langle 1 \rangle = [1]$. Both $f \in C_c^\infty$ and $f = 1$ can seen as functions on the circle $f \in C^\infty(S^1)$. In order to compare the two, we apply Riesz' theorem for functions on the circle, concluding that there is a measure $d\rho$ on S^1 such that for $f \in C^\infty(S^1)$ we have $\langle f \rangle = \int_{S^1} f d\rho = d\rho(\infty) f(\infty) + \int_{-\infty}^\infty f d\rho$. Restricting this to functions with compact support in $\mathbb{R}$ we see that $d\rho|_{\mathbb{R}} = d\mu = \tilde{k} e^{-S} dx$. I.e. for f on the circle $\langle f \rangle = d\rho(\infty) f(\infty) + K[f]$. Taking $f = 1$ gives $1 = d\rho(\infty) + K$, so that for functions on the circle

$$\langle f \rangle = (1 - K) f(\infty) + K[f].$$

Therefore, if f is such that $S'f$ and f' extends to the circle, then $0 = (1 - K)(S'f - f')(\infty)$. To prove that $K = 1$, it suffices to find such a function such that $(S'f - f')(\infty) \neq 0$. Indeed take f to be a C^∞-function which equals $1/S'$ in a neighborhood of ∞. Then $fS' \to 1$ and $f' \to 0$, giving $K = 1$.

3. Next define polynomials by $f_1 := 1$ and $f_{n+1} = F_n . S'$, where $F_n' = f_n$. Then we have $\langle f_n \rangle = [f_n]$: Indeed this is clear for f_1, and by induction we have

$$\langle f_{n+1} \rangle = \langle F_n S' \rangle = \langle f_n \rangle = [f_n] = [F_n S'] = [f_{n+1}].$$

Since the f_n get arbitrarily high degree as n increases, we see that for every $p \in C_{Pol}^\infty$, there is a polynomial $P > p$ such that $\langle P \rangle = [P]$.

4. We now prove the identity $\langle p\rangle = [p]$ for $p \in C^\infty_{Pol}$. By adding polynomials of point 3 to p, we may assume that $p > 0$. We will approximate p by a function p_ϵ of compact support. Then we have
$$|\langle p\rangle - [p]| \leq |\langle p\rangle - \langle p_\epsilon\rangle| + |\langle p_\epsilon\rangle - [p]| = |\langle p\rangle - \langle p_\epsilon\rangle| + |[p_\epsilon] - [p]|.$$
I.e. it suffices to find a p_ϵ such that $|\langle p\rangle - \langle p_\epsilon\rangle|$ and $|[p_\epsilon] - [p]|$ are smaller than ϵ. Indeed by point 3 pick a polynomial $P > p$ such that $[P] = \langle P\rangle$. Since $[.]$ is given by the exponential integral, there is a $P_\epsilon \in C^\infty_c$ such that $0 \leq P_\epsilon \leq P$ and $[P - P_\epsilon] < \epsilon$. And there is a $p_\epsilon \in C^\infty_c$ such that $0 \leq p - p_\epsilon \leq P - P_\epsilon$ and $[p - p_\epsilon] < \epsilon$. Indeed this p_ϵ satisfies
$$|\langle p\rangle - \langle p_\epsilon\rangle| = \langle p - p_\epsilon\rangle \leq \langle P - P_\epsilon\rangle = [P - P_\epsilon] < \epsilon,$$
and
$$|[p_\epsilon] - [p]| = [p - p_\epsilon] < \epsilon.$$

□

A.2.3. *Combinatorical case, normal ordering.* In case the second derivatives of the action can be written as a polynomial of the first derivatives, one may try to solve the Schwinger-Dyson equation by combinatorics, by writing it as
$$\langle \partial_{i_1}(S)...\partial_{i_n}(S)\rangle = \sum_{k=2}^{n} \langle \partial_{i_2}(S)..\partial_{i_1}\partial_{i_k}(S)..\partial_{i_n}(S)\rangle.$$
This way of approaching the Schwinger-Dyson equation was studied in somewhat greater generality in [14]. The normal ordering operation is linked with such combinatorical solutions as follows: If N is invertible, then $\{I$ satisfies the Schwinger-Dyson equation $\Leftrightarrow I(f) = ZN^{-1}(f)I(1)\}$, where Z denotes the projection of polynomial functions of the S_i's on their constant part, e.g. $Z(3 + aS_1 + bS_1S_5) = 3$.

A.2.4. *The Schwinger-Dyson equation for volume manifolds.*

Definition A.2.2. *A volume manifold is a combination (M, μ), where M is a manifold and μ is a volume-form, i.e. a differential form of maximal degree such that μ_m is nonzero in every point $m \in M$. To any manifold is associated an associative algebra A, the real functions, and a Lie algebra L, the vectorfields. In addition to these objects, a volume manifold gives rise to a map $\nabla : L \to A$ defined by $\nabla(X)\mu := L_X\mu$.*

Defining $[f] := \int_M f\mu$, and assuming that $\partial M = 0$, we see that the identity $0 = \int_M L_X(f\mu)$ can be phrased as
$$[X(f)] + [f\nabla(X)] = 0,$$
which is the generalization of the Schwinger-Dyson equation for arbitrary volume manifolds. Thus we see that the statement of the Schwinger-Dyson equation does not need μ, only ∇. In fact:

Theorem A.2.3. *∇ above satisfies the following properties:*
1. *It is closed: $\nabla([X, Y]) = X(\nabla(Y)) - Y(\nabla(X))$.*
2. *It is local: $\nabla(fX) = X(f) + f\nabla(X)$.*
3. *∇ fixes μ up to multiplication by a locally constant function.*

Proof
See [14, Appendix E].

□

We might roughly state the above as follows: If we are not interested in a particular normalization of the integral, then all the information contained in (M, μ) is in (A, ∇). Since furthermore infinite dimensional volume forms do not exist it is useful to define an abstraction of a volumeform which still makes sense in infinite dimensions:

Definition A.2.4. *By a formal volume manifold, we mean a combination* (A, ∇)*, where* A *is a symmetric associative algebra with unit, and* ∇ *is a divergence operator, meaning that with* $L := Der(A)$*,* $\nabla : L \to A$*, such that:*

1. $\nabla([X, Y]) = X(\nabla(Y)) - Y(\nabla(X))$.
2. $\nabla(fX) = X(f) + f\nabla(X)$.

A.3. **Functional integration.** Since finite dimensional integration amounts to finding positive solutions of the finite dimensional Schwinger-Dyson equation, we will define functional integration as finding positive solutions of an infinite dimensional Schwinger-Dyson equation. In this section we will be concerned with stating the infinite dimensional Schwinger-Dyson equation, and will make some remarks on the unicity of their solutions, and conjecture a link with the theory of phase transitions.

A.3.1. *The Schwinger-Dyson equation for linear target spaces.* In this section we will define what we mean by infinite dimensional integration over linear function spaces. To that end we recall the notion of directional differentiation in (possibly infinite dimensional) linear spaces:

Definition A.3.1. *Let* V *be an* $\mathbb{R}$*-vectorspace, let* $v, w \in V$*, and let* $f : V \to \mathbb{R}$*. If it exists, the derivative of* f *at* w *in direction* v *will be defined as*

$$(\partial_v f)(w) := \partial_t|_{t=0} f(w + tv).$$

In finite dimensions we have $(\partial_v f) = v^i \partial_i f$, so that the Schwinger-Dyson equation can be written as $\forall_{v \in \mathbb{R}^n} \quad [(\partial_v S)f] = [\partial_v f]$. This expression can now be generlized to v in infinite dimensional spaces, leading to the following definition:

Definition A.3.2. *Let* M *be a manifold,* $V := Map(M, \mathbb{R}^n)$*, and* $\mathcal{O} := Map(V, \mathbb{R})$ *(observables). Let* $S \in \mathcal{O}$*, and let* C *be a normalization condition (like* $[1] = 1$*), then by a functional integral for* (S, C)*, we mean a linear map on (a subclass* [10] *of)* $\mathcal{O}$*, satisfying:*

1. $f > 0 \Rightarrow [f] > 0$*, (Positivity).*
2. $\forall_{v:M \to \mathbb{R}^n, \text{ compact support}} \ [(\partial_v S)f] = [\partial_v f]$*. (Schwinger-Dyson equation).*
3. $[.]$ *satisfies normalization condition* C*. (Normalization).*

The reason for which we restrict to compact support is that if $\partial M = \emptyset$ then by partial integration this is equivalent to $[(\delta S/\delta\phi(x))f] = [\delta f/\delta\phi(x)]$, which is the usual formulation. Note that if $x \in \partial M$, this last equation will not be true any more. Choosing for the first version makes it possible to derive quantum mechanics from the Schwinger-Dyson equation: This involves functional integrals with an interval $[0, t]$ as a base manifold, and the derivation of the Schroedinger equation (∂_t) involves manipulations at the boundary. Furthermore, we will see in a moment

[10](It is not clear what type of conditions one might need in the future here, but this doesn't seem relevant at this stage of understanding).

that this restriction to compact support seems to be related to the existence of phases and phase transitions.

A.3.2. *Functional integration over nonlinear spaces.* The generalization of the definition A.3.2 to the nonlinear case needs two adaptations: First we have to specify the analogue of the vectorfields that we used, second, we have to work with divergences ∇ instead of actions S.

As vectorfields for the Schwinger-Dyson equation, we propose to use the vectorfields $f \otimes Y$, defined below, where f is a function with compact support on M and Y is a vectorfield on N; The flow of this vectorfield in functionspace is defined as follows: Let $\gamma : M \to N$, then $(F_t^{f\otimes Y}(\gamma)) : M \to N$ is defined by

$$(F_t^{f\otimes Y}(\gamma))(m) := F^Y_{f(m)t}(\gamma(m)).$$

It remains to specify an infinite dimensional divergence ∇. Given the usual identity $\nabla_S = \nabla_0 - dS$ holding for the divergences of the volume forms $\mu_S := e^{-S}\mu_0$, one might think that the only way to specify a divergence ∇_S, given that we are interested in some action S, is to specify ∇_0 (chosen as naturally as possible), and to set $\nabla_S := \nabla_0 - dS$. As explained in the text, this is not the only possibility. A more convenient way to specify ∇_S is by a regulator N of integrands and vectorfields, and setting

$$\nabla_S(N(X)) := -N(X(S)).$$

In this way one performs two tasks at the same time: The specification of the divergence ∇_S, and the specification of UV regulators, which is obligatory for nonlinear targetspaces anyway, since there is no natural distinction between linear and composite integrands in that case.

A.4. **Pure phases, phase regions, phase transitions.**

A.4.1. *An analogy between lattice theories and the Schwinger-Dyson equation with compactly supported vector fields.* In finite dimensions, the sketchy argument of footnote 9 leading to the idea that positivity, normalization and Schwinger-Dyson equation have at most one solution contained the implication $\forall_i\ \partial_i(S-P) = 0 \Rightarrow S = P + c$. Since in infinite dimensions we restrict to vectorfields with compact support, an analogous implication would not be possible since we would have $\forall_v \partial_v(S-P) = 0 \Rightarrow S$ and P differ by a function which only depends on the germ <u>at infinity</u>.

Furthermore, from the work of scientists intersted in the existence of phase transitions in lattice models, see [7, Section 4], [9, section II.5] we know that that different "Gibbs states" may exist for a certain fixed Hamiltonian, these Gibbs states being determined by that Hamiltonian and the conditions <u>at the boundary</u> of an ever-increasing finite subset of the lattice. In that context, one may prove that the set of equilibrium Gibbs states is convex and that each of its points is a convex combination of the extremal points. [11] Pure phases are then defined as these extremal points, the ones in between correspond to states in which these phases coexist.

Having these considerations in mind, we may note that the set of normalized positive solutions of the Schwinger-Dyson equation is a convex set, i.e. if I_1 and I_2

[11] An extremal point of a convex set $\mathcal{C}$ is a point which cannot be written as $ac_1 + (1-a)c_2$ with $c_i \in \mathcal{C}$ and $a \in (0,1)$.

are two such solutions then so is $aI_1 + (1-a)I_2$ for $a \in [0,1]$, and by analogy with the lattice definition of a pure phase, we are led to the following:

A.4.2. *Definition of phase transitions in the context of the Schwinger-Dyson equation.*

Definition A.4.1. *Let $\mathcal{C}_S$ be the set of positive normalized solutions of the Schwinger-Dyson equation for the action S. $\mathcal{C}_S$ is convex.*

1. *By a pure phase of S we understand an extremal point of $\mathcal{C}_S$.*
2. *A one-parameter family $\lambda \mapsto S_\lambda$ of actions is said to undergo a phase transition at λ_0 iff the number of pure phases of S_λ changes at $\lambda = \lambda_0$.*
3. *Two actions are said to be in the same phase region iff they can be joined by a path which does not undergo a phase transition.*

Conjecture A.4.2. *Still by analogy with lattice results: (For more precise formulations see* [9, Section II.5]*)*

1. *Every element of $\mathcal{C}_S$ can be written as a convex combination of extremal elements of $\mathcal{C}_S$. (I.e. extremal points are really part of $\mathcal{C}_S$).*
2. *For a large class of actions S, $\mathcal{C}_{S/\hbar}$ has exactly one point if we take $\hbar$ large enough.*
3. *For a large class of actions S, $\mathcal{C}_S$ has exactly one point if the base manifold is compact.*

References

[1] Feynman, R.P. The space-time approach to non-relativistic quantum mechanics. Reviews of Modern Physics, Vol.20(1948)pp.367-387 and Feynman. Phys. Rev.75(1949)pp. 486-502

[2] Dyson, F. The S-matrix in quantum electrodynamics. Phys.Rev.75(1949) pp. 1736-1755

[3] Houriet, A., Kind, A. Classification invariante des termes de la matrice S. Helvetica Physica Acta 22(1949) pp. 319-330

[4] Schroer, B. The Concept of Nonlocalizable Fields and its Connection with Nonrenormalizable Field Theories. J.Math.Phys. 5(1964) pp. 1361-1367

[5] Streater, R.F.; Wightman, A.S. PCT, Spin and Statistics, and All That. (1964) Benjamin, New York.

[6] Chew, G.F. The analytic S-matrix. (1966) Benjamin, New York.

[7] Dobrushin, R.L. The problem of uniqueness of a Gibbsian random field, and the problem of phase transitions. Functional Analysis and Applications 2(1968)302-312

[8] Wilson, K.G. Non-Lagrangian Models of current algebra. Phys.Rev.179 (1969) pp.1499-1512

[9] Fröhlich, J. The pure phases (harmonic functions) of generalized processes, or Mathematical Physics of Phase transitions and symmetry breaking. Bull.AMS, Vol. 84, Number 1-3 (1978), pp.165-193

[10] Lüscher, M. Solution of the lattice ϕ^4 theory in 4 dimensions. pp.281-307 In: NATO ASI series vol. 185 (Cargese 1987, July 16-30). Nonperturbative quantum field theory. Eds.: 't Hooft; Jaffe, A.; Mack, G.; Mitter, P.K.; Stora, R. Plenum press.

[11] Frenkel, I.B.; Lepowsky, J.; Meurman, A. Vertex operator algebras and the Monster. (1988) Academic, New York.

[12] Segal, G.B. The definition of Conformal Field Theory. In: Differential geometric methods in theoretical physics,pp.165-171, Eds: Bleuler, Werner (1988) Kluwer Academic Publishers.

[13] Atiyah, M.F. Topological Quantum Field Theories. Publ. Math. IHES Paris 68(1988)pp.175-186

[14] de Mirleau, O. Non-Gaussian generalizations of Wick's theorems, related to the Schwinger-Dyson equation. hep-th/9507039.